MULTI-QUARK INTERACTIONS
Effective Lagrangians of Low Energy QCD

- Motivation
- The many-quark vertices Lagrangian
- Bosonization
- Functional integration methods and physical implications

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Powerful classification schemes in low-energy QCD:

In absence of a quantitative framework within QCD to deal with its large distance dynamics, one recurs to phenomenological descriptions subject to

- **symmetries and symmetry breaking schemes**
- **large N_c counting rules**
- **critical mass scales at which some type of interactions set in**
- **hierarchy in the interactions**

A well known example: η' meson physics

Lightest observed $SU(3)$ singlet pseudoscalar:
$m_{\eta'} \sim 1 GeV$.

Above concepts needed to understand its large mass.

A brief History:

If singlet $SU(3)$ axial current of QCD, $U_A(1)$, conserved as $m_{curr} \longrightarrow 0$
$\longrightarrow \longrightarrow \eta'$ as 9th pseudoscalar Goldstone.

1969:the global $U_L(3) \times U_R(3)$ chiral symmetry of QCD is broken by the $U_A(1)$ Adler-Bell-Jackiw anomaly

1976: study of instantons by 't Hooft $\longrightarrow$

1978:effective $2N_f$ quark vertices, the 't Hooft interactions, responsible for the non-conservation of the singlet axial current and absence of the related Goldstone boson.

1979:Wittens' puzzle.

Typically, the mass squared of an approximate Goldstone boson is linear in the symmetry breaking parameter. The anomaly is order $1/N_c$, one expects $m^2_{\eta'}$ to be of order $(1/N_c)$.
Although the $U_L(3) \times U_R(3)$ symmetry is recovered in the limit $N_c \rightarrow \infty$, the empirical mass of the η' is abnormaly large to conform alone with the large N_c counting rules.

• 1981:Solution of Witten's puzzle by Novikov, Shifman, Vainshtein, Zakharov.

With the help of QCD sum rules it was shown that a large scale, or critical mass $M_{crit}^2 \simeq 4.2 - 6.6\mathrm{GeV}^2$emerges in the $0^-, 0^+$ gluonic channels, characterizing the breaking of asymptotic feedomin these channels.

The big mass of η' was explained through mixing with glue states, together with the existence of a natural scale compared to which the η' mass is small.

It became at the same time clear that the accuracy of the $1/N_c$ series gets worse in the presence of large critical masses,i.e. the pictures at $N_c \to \infty$ and at $N_c = 3$ are qualitatively different from each other in these cases.

• P. Di Vecchia, Phys. Lett. B 85 (1979) 357; P. Di Vecchia and G. Veneziano, Nucl. Phys. B 171 (1980) 253; Effective Lagrangian which includes the meson fields and the topological charge density $Q(x)$. Eliminating the field $Q(x)$ by means of its classical equation of motion, one obtains an effective mesonic Lagrangian.

• Rosenzweig, Schechter and Trahern, Phys. Rev. D 21 (1980) 3388.
the 't Hooft type determinantal interaction, written in terms of mesonic fields, appears as the first term in the expansion which results from eliminating $Q(x)$.

• One realizes again: there are no valid theoretical objections against the idea that the 't Hooft interaction and higher order multi-quark terms are actually present in the QCD vacuum.

Our plan and motivation

• We address the η' physics and lowest lying $(0^-, 0^+)$ meson nonets in a many-fermion vertices Lagrangian without explicit gluon degrees of freedom, with underlying symmetries of QCD.

• Semiclassical theory based on the QCD instanton vacuum provides evidence in favor of $2N_f$-quark interactions in low energy QCD. In leading $1/N_c$ they are given by the 't Hooft determinant, which breaks the axial $U_A(1)$ symmetry and is the source of OZI-violating effects.
G. 't Hooft,Phys.Rev. D14 3432 (1976); Phys.Rev. D18 2199 (1978).

• The Effective Quark Lagrangian derived from the instanton-gas model predicts the existence of 4q, 6q, ...nq ($q \equiv quark$) interactions in the large N_c limit, all equally weighted. The 't Hooft type Ansatz emerges if only zero modes contribute.
• Yu. A. Simonov, Phys.Lett. B412 371 (1997). • Phys.Rev. D65 094018 (2002)

• However accurate lattice measurements for the realistic QCD vacuum show hierarchy between gluon field correlators with dominance of lowest one.
• G.S. Bali, Phys. Rev. D62 114503 (2000).

• It can trigger a similar hierarchy in multi-quark intearctions after integrating out the gluon degrees of freedom.

Conclusions

- The bosonized 4-quark NJL together with the 't Hooft six-quark flavor determinant Lagrangian, NJLH, has a fatal flaw: it has no stable vacuum.

- The hierarchy hypothesis allows to treat the 't Hooft interaction perturbatively around the stable NJL Lagrangian.

- Known analogy: the harmonic oscilator together with a cubic driving term is not a stable system. However the cubic term can be treated perturbatively and leads to a well defined spectrum.

- The perturbative series can be resummed to yield a Loop expansion.

- Uncovering of new contributions, albeit singular, to the phase (even number of loops) and measure (odd number of loops) of the functional integral. Classification of these contributions through Feynman diagrams.

- Singularities emerge from the local character of the interactions. They can be removed taking non-local vertices. Presently there are many efforts to implement non-locality.

- Since even local interactions are already quite complicated calculations, the singularities can be treated in a first approach as effective parameters, together with the cutoff of the quark loop diagrmas.

• The global instability of the vacuum can be removed within the multi-quark interaction picture: the addition of a chiral invariant OZI-violating, Okubo:1963, eight-quark interaction to the Lagrangian renders it globally stable.

• A comparison between the effective potentials for the NJLH and for the eight-quark enlarged Lagrangian reveals that the global stabilization happens around the local minimum which appears if only the regular critical point is considerd in the NJLH Lagrangian. This is probably at the heart of its success.

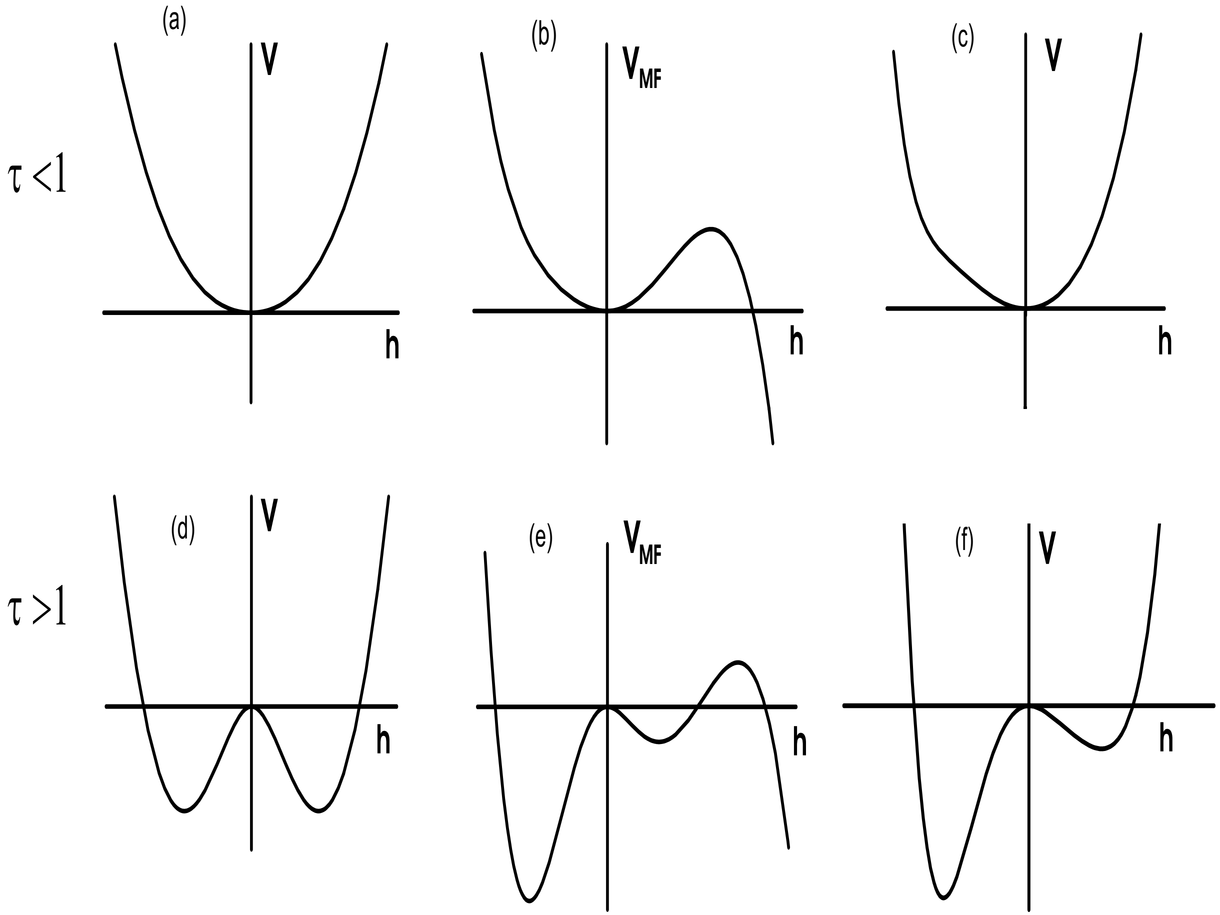

Figure 1: Effective Potential

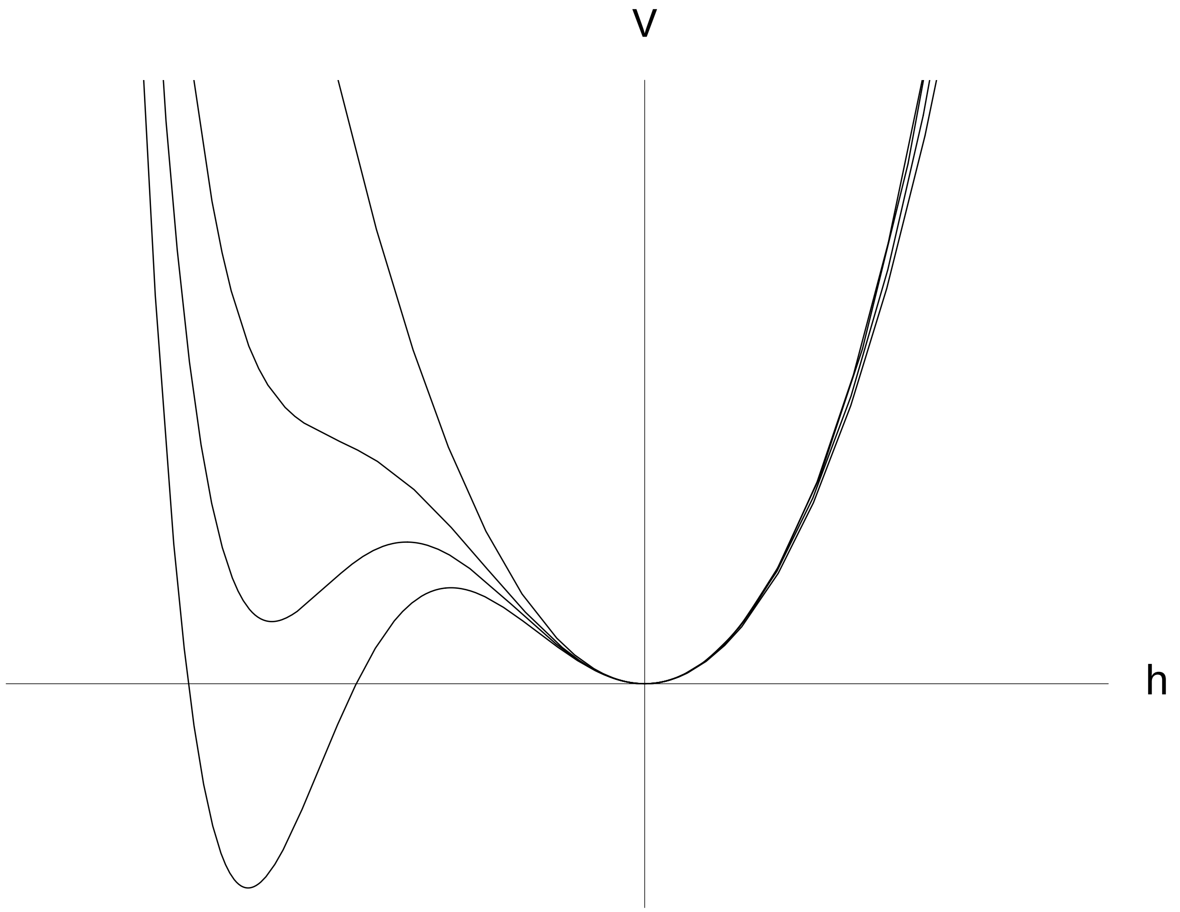

Figure 2: Effective Potential

The effective quark model Lagrangian

- **Hierarchy assumption:**

i) - 4-fermion vertices most important (NJL)

- Y. Nambu, G. Jona-Lasinio, Phys.Rev.122, 345(1961); Phys.Rev.124,246(1961);

adapted to the $U_L(3) \times U_R(3)$ chiral symmetry of massless QCD.

ii) - Join with the next in the hierarchy 't Hooft 6-fermion interactions.

$$\mathcal{L} = \bar{q}(i\gamma^{\mu}\partial_{\mu} - \hat{m})q + \mathcal{L}_{NJL} + \mathcal{L}_6, \tag{1}$$

NJL four-fermion vertices of the scalar and pseudoscalar types

$$\mathcal{L}_{NJL} = \frac{G}{2}[(\bar{q}\lambda_a q)^2 + (\bar{q}i\gamma_5\lambda_a q)^2] \tag{2}$$

Six-quark 't Hooft interaction

$$\mathcal{L}_6 = \kappa(\det\, \bar{q}P_L q + \det\, \bar{q}P_R q), \tag{3}$$

- Positive coupling G, $[G] = \mathrm{GeV}^{-2}$ has order $G \sim 1/N_c$

- Negative coupling κ, $[\kappa] = \mathrm{GeV}^{-5}$ with large N_c asymptotics $\kappa \sim 1/N_c^{N_f}$.

- $P_{L,R} = (1 \mp \gamma_5)/2$ are projectors and the determinant is over flavor indices.

- $\mathcal{L}_{NJL}$ dominates over $\mathcal{L}_6$ at large N_c.

Model explored widely at mean field level. Pioneering works:

- V.Bernard, R.L. Jaffe, U.-G. Meißner, Nucl.Phys. B308, 753 (1988);
- H.Reinhardt, R.Alkofer,Phys.Lett. B198, 92 (1987).

Bosonization

Vacuum persistance amplitude

$$Z = \int \mathcal{D}q\mathcal{D}\bar{q} \exp\left(i \int d^4x \mathcal{L}\right). \tag{4}$$

Many fermion vertices of the original Lagrangian linearized with functional identity

$$\begin{aligned} 1 &= \int \prod_a \mathcal{D}s_a \mathcal{D}p_a \delta(s_a - \bar{q}\lambda_a q)\delta(p_a - \bar{q}i\gamma_5\lambda_a q) \\ &= \int \prod_a \mathcal{D}s_a \mathcal{D}p_a \mathcal{D}\sigma_a \mathcal{D}\phi_a \\ &\quad \times \exp\left\{ i \int \mathrm{d}^4x [\sigma_a(s_a - \bar{q}\lambda_a q) + \phi_a(p_a - \bar{q}i\gamma_5\lambda_a q)] \right\}, \end{aligned} \tag{5}$$

$$
\begin{aligned}
Z \;=\; & \int \prod_a \mathcal{D}\sigma_a \mathcal{D}\phi_a \mathcal{D}q \mathcal{D}\bar{q} \exp\left(i \int d^4x \mathcal{L}_q(\bar{q}, q, \sigma, \phi) \right) \\
& \int \prod_a \mathcal{D}s_a \mathcal{D}p_a \exp\left(i \int d^4x \mathcal{L}_r(\sigma, \phi, s, p) \right)
\end{aligned}
\tag{6}
$$

Here $\sigma = \sigma_a \lambda_a$, etc.

$$
\mathcal{L}_q(\bar{q}, q, \sigma, \phi) \;=\; \bar{q}(i\gamma^\mu \partial_\mu - \sigma - i\gamma_5 \phi) q, \tag{7}
$$

$$
\begin{aligned}
\mathcal{L}_r(\sigma, \phi, s, p) \;=\; & \frac{G}{2} \left[(s_a)^2 + (p_a)^2 \right] + s_a(\sigma_a - \hat{m}_a) + p_a \phi_a \\
+ \; & \frac{\kappa}{32} A_{abc} s_a \left(s_b s_c - 3 p_b p_c \right).
\end{aligned}
\tag{8}
$$

$\mathcal{L}_r$ is a functional integral with a cubic polynomial in the exponent.

The totally symmetric constants A_{abc} are related to the flavour determinant, and equal to

$$
A_{abc} \;=\; \frac{1}{3!} \epsilon_{ijk} \epsilon_{mnl} (\lambda_a)_{im} (\lambda_b)_{jn} (\lambda_c)_{kl} \tag{9}
$$

•Scalar fileds $\sigma_a,\ a = 0, 3, 8$ have non-zero vacuum expectation values related to spontaneous breaking of global chiral symmetry, which give rise to the constituent masses m_a of the quarks.

New scalar fields with zero vacuum expectation values $< 0|\sigma_a|0 >= 0$ are defined through the shift $\sigma_a \longrightarrow \sigma_a + m_a$.

• We seek the final bosonized Lagrangian expressed in terms of the mesonic fields $\sigma_a$and ϕ_a.

• Integrate out all remaining fields.

• Convenient notation:
Join the auxiliary bosonic variables in one object

$R_A = (R_a, R_{\dot{a}})$ with $R_a \equiv s_a$ and $R_{\dot{a}} \equiv p_a$.
Then $R_A^2 = s_a^2 + p_a^2$.
Analogously, use $\Pi_A = (\sigma_a, \phi_a)$ for external fields and $\Delta_A = (\Delta_a, 0)$.

Let

$$\Phi_{abc} = \frac{3}{16} A_{abc}, \quad \Phi_{a\dot{b}\dot{c}} = -\frac{3}{16} A_{abc}, \quad \Phi_{ab\dot{c}} = 0, \quad \Phi_{\dot{a}\dot{b}\dot{c}} = 0. \tag{10}$$

$$\frac{\kappa}{32} A_{abc} s_a (s_b s_c - 3 p_b p_c) = \frac{\kappa}{3!}\, \Phi_{ABC} R_A R_B R_C \tag{11}$$

Important property to be fulfilled

$$\Phi_{ABC}\delta_{BC} = 0. \tag{12}$$

Functional integral over auxiliary fields R_A can be written in the compact way

$$\mathcal{Z}[\Pi, \Delta] \equiv \mathcal{N} \int_{-\infty}^{+\infty} \prod_A \mathcal{D}R_A \ \exp\left(i \int \mathrm{d}^4x \mathcal{L}_r(\Pi, \Delta; R) \right), \tag{13}$$

where

$$\mathcal{L}_r = R_A(\Pi_A + \Delta_A) + \frac{G}{2} R_A^2 + \frac{\kappa}{3!} \Phi_{ABC} R_A R_B R_C. \tag{14}$$

METHODS OF SOLUTION

The integration over the auxiliary variables R_A

Cubic structure: temptet to solve it using Airy's integral methods.
Large expansion parameter, N_c: one uses the well known asymptotics of the Airy's function on the real axis. This requires

$$\frac{\partial^2 \mathcal{L}_r}{\partial R_A \partial R_B} \equiv \mathcal{L}''_{AB} = G\delta_{AB} + \kappa \Phi_{ABC} R_C = 0 \tag{15}$$

which cannot be fulfilled.

1. STATIONARY PHASE METHOD and a PROBLEM

Suppose that both interactions, $\mathcal{L}_{\rm NJL}$ and $\mathcal{L}_6$, are equally weighted.

- To obtain the semiclassical asymptotics use the stationary phase approximation (SPA). All critical points which solve the system of equations for the first order derivatives

$$\frac{\partial \mathcal{L}_r}{\partial R_A} = G R_A + \Delta_A + \Pi_A + \frac{\kappa}{2}\, \Phi_{ABC} R_B R_C = 0 \tag{16}$$

must be taken into account.

- Solutions of eq.(16) can be obtained selfconsistently.
- A.A. Osipov, B. Hiller, EPJC 35, 223, (2004).

Up to some order in powers of external mesonic fields, Π_A, get polynomial

$$\mathcal{R}_A^{(i)} = H_A^{(i)} + H_{AB}^{(i)} \Pi_B + H_{ABC}^{(i)} \Pi_B \Pi_C + H_{ABCD}^{(i)} \Pi_B \Pi_C \Pi_D + \ldots \tag{17}$$

where $i = 1, 2, \ldots$ indicate different possible solutions and coefficients $H_{A\ldots}^{(i)}$ depend on Δ_a, G, κ.

• Put this expansion in eq.(16) and obtain a series of self-consistent equations to determine $H^{(i)}_{A\ldots}$.

The first one is

$$G H_A^{(i)} + \Delta_A + \frac{\kappa}{2}\,\Phi_{ABC} H_B^{(i)} H_C^{(i)} = 0. \tag{18}$$

i) Trivial solution $H_A = 0$, corresponding to the unbroken vacuum $\Delta_A = 0$.

ii) Non-trivial solutions for the scalar component, i.e.,

$$H_A^{(i)} = (h_a^{(i)}, 0). \tag{19}$$

The number of possible solutions, i, depends on the symmetry group. Pattern of explicit symmetry breaking $\longrightarrow$
mean field can have only three non-zero components at most, with indices $a = 0, 3, 8$.
$h^{(i)} = h_a^{(i)} \lambda_a = \mathrm{diag}(h_u^{(i)}, h_d^{(i)}, h_s^{(i)})$

$$\begin{cases} G h_u + \Delta_u + \frac{\kappa}{16} h_d h_s = 0 \\ G h_d + \Delta_d + \frac{\kappa}{16} h_s h_u = 0 \\ G h_s + \Delta_s + \frac{\kappa}{16} h_u h_d = 0 \ . \end{cases} \tag{20}$$

This system is equivalent to a fifth order equation for a one-type variable which can be solved numerically.

Analytical solutions for:
i) $\hat{m}_u = \hat{m}_d = \hat{m}_s,$ $SU(3)$ limit
ii)$\hat{m}_u = \hat{m}_d \neq \hat{m}_s,$ $SU(2)_I \times U(1)_Y$

The higher index coefficients $H^{(i)}_{A\dots}$ are recurrently expressed by means of lower ones. For instance we have

$$\left(H^{(i)}_{AB}\right)^{-1} = -\left(G\delta_{AB} + \kappa \Phi_{ABC} H^{(i)}_C\right) , \tag{21}$$

$$H^{(i)}_{ABC} = \frac{\kappa}{2} \Phi_{DEF} H^{(i)}_{DA} H^{(i)}_{EB} H^{(i)}_{FC} \ , \tag{22}$$

and so on.

$SU(3)$ case:

$$h_u^{(1)} = -\frac{8G}{\kappa}\left(1-\sqrt{1-\frac{\kappa\Delta_u}{4G^2}}\right) , \qquad h_u^{(2)} = -\frac{8G}{\kappa}\left(1+\sqrt{1-\frac{\kappa\Delta_u}{4G^2}}\right) \tag{23}$$

$SU(2)_I \times U(1)_Y$

$$h_u^{(1)}=2\sqrt{-Q}\sin\left(\frac{\phi}{3}-\frac{\pi}{6}\right), \qquad h_u^{(2)} = -2\sqrt{-Q}\sin\left(\frac{\phi}{3}+\frac{\pi}{6}\right),$$
$$h_u^{(3)}=2\sqrt{-Q}\cos\frac{\phi}{3}, \quad h_d^{(i)} = h_u^{(i)}, \quad h_s^{(i)} = -\frac{\Delta_s}{G} - \frac{\kappa}{16G}\left(h_u^{(i)}\right)^2 \tag{24}$$

$$\cos\phi = \frac{R}{\sqrt{-Q^3}} , \quad \sin\phi = \sqrt{1+\frac{R^2}{Q^3}} , \quad 0 \le \phi \le \pi , \tag{25}$$

$$Q = \left(\frac{16G}{\kappa}\right)^2 \frac{x_s-1}{3} , \quad R = \left(\frac{16G}{\kappa}\right)^3 \frac{x_u}{2} , \quad x_f = \frac{\kappa\Delta_f}{(4G)^2} . \tag{26}$$

The lowest order semiclassical asymptotics

Replace $R_A \to \bar{R}_A = R_A - \mathcal{R}_A^{(i)}$ in the functional integral (13) to obtain the semiclassical asymptotics

$$\begin{aligned}\mathcal{Z}[\Pi,\Delta] \sim \mathcal{N} \sum_{i=1}^{n} & \exp\left(i\int \mathrm{d}^4x \mathcal{L}_{\mathrm{st}}^{(i)}\right) \\ & \times \int_{-\infty}^{+\infty} \prod_A \mathcal{D}\bar{R}_A \exp\left(\frac{i}{2}\int \mathrm{d}^4x \mathcal{L}''_{AB}(\mathcal{R}^{(i)})\bar{R}_A\bar{R}_B\right) \\ & \times \sum_{k=0}^{\infty}\left(i\frac{\kappa}{3!}\Phi_{ABC}\int \mathrm{d}^4x \bar{R}_A\bar{R}_B\bar{R}_C\right)^k \qquad (\hbar \to 0) \end{aligned} \tag{27}$$

where n is the number of solutions, $\mathcal{R}_A^{(i)}$.

$$\begin{aligned}\mathcal{L}_{\mathrm{st}}^{(i)} &= \mathcal{R}_A^{(i)}(\Pi_A + \Delta_A) + \frac{G}{2}\mathcal{R}_A^{(i)2} + \frac{\kappa}{3!}\Phi_{ABC}\mathcal{R}_A^{(i)}\mathcal{R}_B^{(i)}\mathcal{R}_C^{(i)} \\ &= \frac{G}{6}\left(\mathcal{R}_A^{(i)}\right)^2 + \frac{2}{3}\mathcal{R}_A^{(i)}\Pi_A = h_a^{(i)}\sigma_a + \mathcal{O}(\Pi^2). \end{aligned} \tag{28}$$

The linear term in the σ field is responsible for the dynamical symmetry breaking in the multi-quark system. Taken together with the corresponding part from the Gaussian integration over quark fields leads to the gap equations.

At leading order, $k = 0$, we get
$SU(3)$ limit:

$$\mathcal{Z} \sim \exp\left(i \int \mathrm{d}^4 x \sum_{i=1}^{2} h_a^{(i)} \sigma_a + \ldots \right)$$
$$\sim \exp\left(-i \frac{8G}{\kappa} \int \mathrm{d}^4 x (\sigma_u + \sigma_d + \sigma_s) + \ldots \right) \tag{29}$$

$SU(2)_I \times U_Y(1)$:

$$\mathcal{Z} \sim \exp\left\{ \frac{i}{2} \int \mathrm{d}^4 x \sum_{i=1}^{3} \left(h_u^{(i)} (\sigma_u + \sigma_d) + h_s^{(i)} \sigma_s \right) \right\}$$
$$\sim \exp\left\{ -\frac{i}{2} \left(\frac{\Delta_s}{G} + \frac{32G}{\kappa} \right) \int \mathrm{d}^4 x \, \sigma_s + \ldots \right\} \tag{30}$$

• In both cases the contribution has positive sign.
Since the quark loop also contributes with a positive sign, there are no solutions for gap equations at leading order.
Conclusion: The considered model can be used only in the framework of the perturbative approach. It assumes the hierarchy of multi-quark interactions.

Effective potential for $SU(3)$ limit:

$$V(M) = \frac{h^2}{16}\left(12G + \kappa h + \frac{27}{2}\lambda h^2\right) - \frac{3N_c}{16\pi^2}\left[M^2 J_0(M^2) + \Lambda^4 \ln\left(1 + \frac{M^2}{\Lambda^2}\right)\right], \tag{31}$$

with Λ being an ultraviolet cutoff in the quark one-loop diagrams

$$J_0(M^2) = \Lambda^2 - M^2 \ln\left(1 + \frac{\Lambda^2}{M^2}\right). \tag{32}$$

The dependence on the variable h is defined by the stationary phase equation

$$M + Gh + \frac{\kappa}{16}h^2 + \frac{3}{4}\lambda h^3 = 0, \qquad \lambda \equiv g_1 + \frac{2}{3}g_2 \tag{33}$$

The perturbative approach

- The free part,

$$\mathcal{L}_0(R_A) = \frac{G}{2} R_A^2 + R_A(\Pi_A + \Delta_A). \tag{34}$$

- and the 't Hooft interaction piece, considered as a perturbation

$$\mathcal{L}_I(R_A) = \frac{\kappa}{3!} \Phi_{ABC} R_A R_B R_C. \tag{35}$$

- Perturbative representation for the functional integral

$$\mathcal{Z} = N' \exp\left(i \int \mathcal{L}_I(\hat{X}_A)\right) \int \prod_A \mathcal{D}R_A \, e^{i \int \mathcal{L}_0(R_A)} \tag{36}$$

where

$$\hat{X}_A = -i \frac{\delta}{\delta \Pi_A}. \tag{37}$$

- Integrate out boson fields (appear quadratically)

$$\mathcal{Z} = N \exp\left(i \int \mathcal{L}_I(\hat{X})\right) \exp\left(-i \int \frac{\bar{\Pi}_A^2}{2G}\right) \tag{38}$$

where $\bar{\Pi}_A = \Pi_A + \Delta_A$.

• By definition the effective action $\Gamma_{\rm eff}$ is the phase of $\mathcal{Z}$

$$\mathcal{Z} = A(\bar{\Pi}_A)\,\exp\left(i\int \mathcal{L}_{\rm eff}(\bar{\Pi}_A)\right), \tag{39}$$

and $A(\bar{\Pi}_A)$ is a real function.

$$\Gamma_{\rm eff} = i\ln\frac{A}{N} + \Gamma_0 - i\ln\left(1 + e^{-i\Gamma_0}\left(e^{i\int \mathcal{L}_I} - 1\right)e^{i\Gamma_0}\right). \tag{40}$$

Γ_0 : the leading order action

$$\Gamma_0 = -\frac{1}{2G}\int \bar{\Pi}_A^2. \tag{41}$$

• Source of $U(1)$ breaking corrections (2nd log) arise as a series in powers of the functional derivatives operator

$$\hat{\Gamma}_I = \int \mathcal{L}_I(\hat{X}_A) = \frac{\kappa}{3!}\Phi_{ABC}\int \hat{X}_A\hat{X}_B\hat{X}_C\,. \tag{42}$$

• Consider the expansion

$$\delta = e^{-i\Gamma_0}\left(e^{i\hat{\Gamma}_I} - 1\right)e^{i\Gamma_0} = \sum_{m=1}^{\infty}\frac{i^m}{m!}\left(e^{-i\Gamma_0}\hat{\Gamma}_I e^{i\Gamma_0}\right)^m. \tag{43}$$

$$e^{-i\Gamma_0}\hat{\Gamma}_I e^{i\Gamma_0} = -\frac{\kappa}{3!}\,\Phi_{ABC}\int\left(\frac{1}{G^3}\,\bar{\Pi}_A\bar{\Pi}_B\bar{\Pi}_C - \frac{3}{G^2}\,\bar{\Pi}_A\bar{\Pi}_B\hat{X}_C + \frac{3}{G}\,\bar{\Pi}_A\hat{X}_B\hat{X}_C - \hat{X}_A\hat{X}_B\hat{X}_C\right). \tag{44}$$

- Represent the effective action (40) as a perturbative series in powers of κ,

$$\delta = \sum_{n=1}^{\infty}\kappa^n\delta_n, \tag{45}$$

- To the second order in κ we have

$$\Gamma_{\text{eff}} = i\ln\frac{A}{N} + \Gamma_0 - i\kappa\delta_1 - i\kappa^2\left(\delta_2 - \frac{1}{2}\delta_1^2\right) - \ldots \tag{46}$$

where

$$\delta_1 = \frac{-i}{3!G^3}\,\Phi_{ABC}\int\bar{\Pi}_A\bar{\Pi}_B\bar{\Pi}_C\,, \tag{47}$$

$$\delta_2 - \frac{\delta_1^2}{2} = -\frac{i}{8G^5}\,\Phi_{ABC}\Phi_{AEF}\int\bar{\Pi}_B\bar{\Pi}_C\bar{\Pi}_E\bar{\Pi}_F + \delta(0)\int\frac{\bar{\Pi}_A^2}{(8G^2)^2} + [\delta(0)]^2\int\frac{3i}{32G^3}\,. \tag{48}$$

• The real factor $A(\bar{\Pi})$
is the measure of the functional integral at integration over σ_a, ϕ_a.

The model infinities

•The perturbative approach is essentially based on the leading role of the fundamental four-quark interactions. So it inherits all shortcomings known from the NJL model:

•it is nonrenormalizable and fundamental interactions must be cutoff. We view the cutoff as an effective, if crude, implementation of the known short distance behaviour of QCD within the model.

•The $\delta(0)$-singularities, found above, are caused by the local structure of the last exponent in eq.(38). If one would start with nonlocal NJL interactions, the divergencies could be lower or would even disappear.

•Origin of singularities understood with Feynman diagrams.

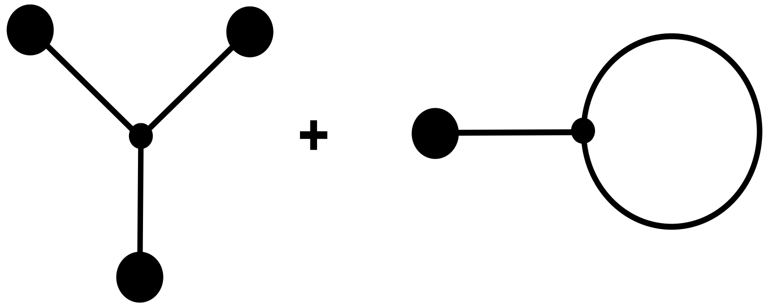

Figure 3: Lowest order in κ graphs, δ_1

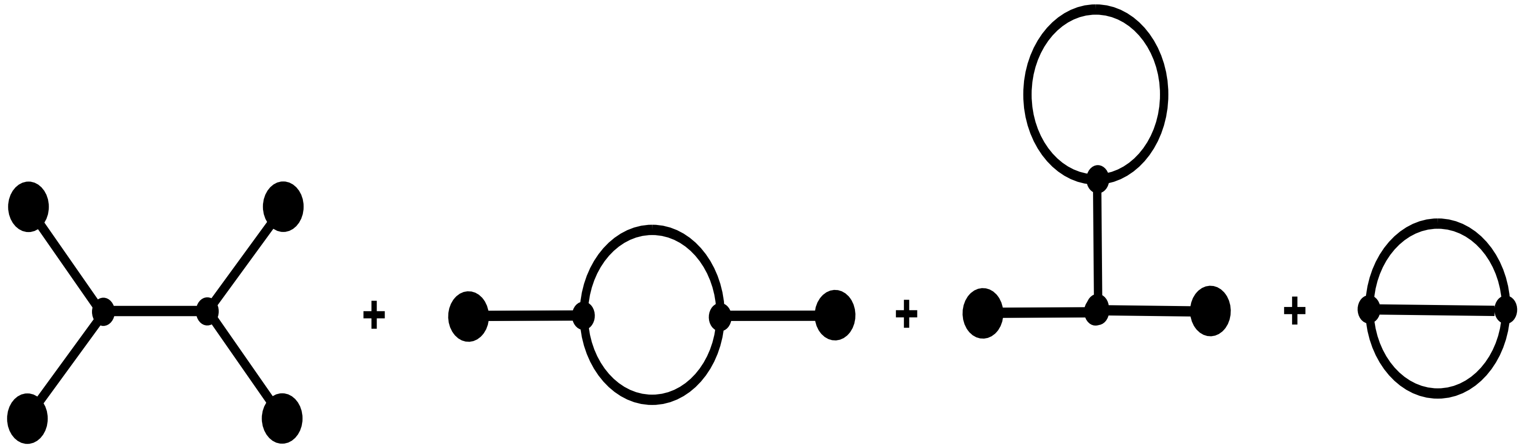

Figure 4: Order κ^2 graphs; δ_2

For any diagram

$$E = 2I - 3V$$
$$L = I - V - E + 1\,. \tag{49}$$

E: number of external fields,, $i\int \mathrm{d}^4x\bar{\Pi}_A(x)$
I: the number of internal lines,
V: number of vertices, $i\kappa\Phi_{ABC}\int \mathrm{d}^4x$
Internal line: "propagator" of the form

$$\Delta_{AB}(x-y) = -(i/G)\delta_{AB}\delta(x-y). \tag{50}$$

L: number of loops,
• The number of loops in a diagram is equal to the number of δ functions surviving after integration over coordinates.

• Every internal line contributes one δ function.
• But every vertex or external field gives an integration over the corresponding coordinate, reducing by one the number of δ functions.
• One integration is kept for over-all integration of the effective action.

• One-loop tadpoles add up to zero.

A by-pass road to sum the perturbative series – LOOP EXPANSION

Introducing a parameter t in the perturbative representation (36)

$$G, \kappa, \Pi_A, \Delta_A \rightarrow tG, t\kappa, t\Pi_A, t\Delta_A. \tag{51}$$

$$\Gamma_{\text{eff}} \rightarrow t\Gamma'_{\text{eff}}(t) \tag{52}$$

Groups in terms of inverse powers of t of $\Gamma'_{\text{eff}}(t)$ define the new series:

- No closed loops (tree graphs) will carry the weight 1.
- All the one-loop diagrams (and higher odd numbers of loop diagrams) contribute to the imaginary part of the action. They will be

cancelled by the appropriate choice of the real quantity $\frac{1}{t}A(\bar{\Pi}_A)$, with A being of order 1. They contribute to the measure of the integral over Π_A.

• The set of graphs with $2n$ loops have the factor t^{-2n} and contribute to the effective action; in general they include, as a subset, all graphs of k^{n+1}th order or higher in this coupling constant. The resummed series is equivalent to the well-known loop expansion of Coleman 1973. The latter can be written formally as:

Go back to the SPA case:

$$\mathcal{Z}[\Pi,\Delta]\sim\mathcal{N}\sum_{i=1}^{n}\ \exp\left(i\int \mathrm{d}^4x\mathcal{L}_{\rm st}^{(i)}\right)$$
$$\times\int_{-\infty}^{+\infty}\prod_A \mathcal{D}\bar{R}_A\ \exp\left(\frac{i}{2}\int \mathrm{d}^4x\mathcal{L}''_{AB}(\mathcal{R}^{(i)})\bar{R}_A\bar{R}_B\right)$$
$$\times\sum_{k=0}^{\infty}\left(i\frac{\kappa}{3!}\ \Phi_{ABC}\int \mathrm{d}^4x\bar{R}_A\bar{R}_B\bar{R}_C\right)^k \qquad (\hbar\longrightarrow 0)\ (53)$$

and take only one of the n solutions, corresponding to the minimum.

• We are able to show that this result is fully equivalent to the perturbative approach, when expanded up to the order κ considered in the perturbation.

• Each term of the k-sum can be integrated in closed form. The zero-th order contribution yields, up to the pre-exponential factor I_0

the result of H. Reinhardt and R. Alkofer Phys.Lett. B198, 92 (1987).

$$\mathcal{Z}[\Pi, \Delta] \sim I_0 \exp(i \int \mathrm{d}^4 x \mathcal{L}_{\mathrm{st}}^{(i)}) \tag{54}$$

$$I_0 = \frac{1}{\sqrt{det \mathcal{L}''}} (\sqrt{\frac{2\pi}{V}})^N exp(i \frac{\pi}{4} \sum_{a=1}^{N} sgn(\lambda_a)) \tag{55}$$

with N=18 (the number of fields in the two nonets) and λ_a the eigenvalues resulting from the integration over the quadratic integration involving $\mathcal{L}''_{AB}$. Here V is a small volume associated with the discretization $\int d^4x = V \sum_x$. In each cell V the fields are supposed not to change.

In A.A. Osipov, B. Hiller, EPJC 35, 223, (2004)we have calculated the preexponential factor $I_0 \sim \delta(0)$ which contributes to the measure in the Π functional integration .

- The term k=1 vanishes (integration over an odd number of fields).
- The term k=2 yields all $\delta(0)^2$ proportional terms and are an essential contribution to the phase (effective action).

We obtain to order k=2:

$$\mathcal{L}_{eff} = \mathcal{L}_{st} + \frac{3\kappa^2\delta(0)^2}{32N(N+2)(N+4)}[(tr\mathcal{L}''^{-1})^3 + 6tr(\mathcal{L}''^{-1})tr(\mathcal{L}''^{-1})^2 + 8tr(\mathcal{L}''^{-1})^3] \quad (56)$$

Dimensionless expansion parameter

Estimates to justify the result. Neglect symmetry group, discretize spacetime

$$\mathcal{Z}[\Pi,\Delta] \sim \prod_x \int \mathrm{d}R_x \exp\left\{ i\Omega\left(\mathcal{L}_{\mathrm{st}} + \frac{1}{2}\mathcal{L}''_{\mathrm{st}} R_x^2 + \frac{1}{3!}\mathcal{L}'''_{\mathrm{st}} R_x^3 \right)\right\} . \tag{57}$$

Assume that

$$\Omega \mathcal{L}_{\mathrm{st}} \gg 1. \tag{58}$$

Dominating role of the Gaussian integral if essential values for R_x in the integral have the order $R_x^2 \sim 1/(\Omega \mathcal{L}''_{\mathrm{st}})$. For cubic term

$$\Omega \mathcal{L}'''_{\mathrm{st}} R_x^3 \sim \sqrt{\frac{(\mathcal{L}'''_{\mathrm{st}})^2}{\Omega (\mathcal{L}''_{\mathrm{st}})^3}} \sim \sqrt{\frac{\kappa^2}{\Omega G^3}} \sim \sqrt{\zeta} \, , \tag{59}$$

Here $\mathcal{L}'''_{\mathrm{st}} \sim \kappa$, $\mathcal{L}''_{\mathrm{st}} \sim G$. If the parameters of the model chosen such $\zeta \ll 1$ is fulfilled, the cubic power of R_x yields terms that go to zero relative to the Gaussian term as $\zeta \to 0$, and the stationary phase approximation will be justified.

Ω may be written as an ultraviolet divergent integral regularized by introducing a cutoff Λ

$$\Omega^{-1} = \delta^4(0) \sim \int_{-\Lambda_1/2}^{\Lambda_1/2} \frac{\mathrm{d}^4 k_e}{(2\pi)^4} = \left(\frac{\Lambda_1}{2\pi}\right)^4 . \tag{60}$$

Dimensionless parameter

$$\zeta = \frac{\kappa^2}{32G^3}\left(\frac{\Lambda_1}{2\pi}\right)^4 . \tag{61}$$

• A.A. Osipov, Brigitte Hiller, Joao Moreira, Alex H. Blin, Eur.Phys.J.C (2006), hep-ph/0601074

Without two-loop, $\lambda = 0$:
•A.A. Osipov, H. Hansen, B. Hiller, Nucl.Phys.A745,81,2004

Table 1.
The main parameters of the model: current quark masses $\hat{m}_u$, $\hat{m}_s$, and corresponding constituent masses, m_u and m_s in MeV, couplings G (in GeV^{-2}) and κ (in GeV^{-5}), and two cutoffs Λ, λ in GeV. The values of condensates are given in MeV.

	$\hat{m}_u$	$\hat{m}_s$	m_u	m_s	G	$-\kappa$	Λ	λ	$-\langle\bar{u}u\rangle^{1/3}$	$-\langle\bar{s}s\rangle^{1/3}$
a	6.3	194	398	588	13.5	1300*	0.82	0*	229	172
b	6.3	194	398	588	13.5	1300*	0.82	1.8*	229	172
c	6.3	194	398	588	13.4	1370*	0.82	0*	229	172
d	6.3	194	398	588	11.8	1370*	0.82	1.7*	229	172
e	2.8	92	216	385	3.14	120	1.37	0*	302	314
f	2.1	69	196	354	2.15	53	1.64	1.9*	333	363

Table 2.
The main characteristics of the light pseudoscalar mesons in MeV. The singlet-octet mixing angle θ_p is given in degrees.

	m_π	m_K	f_π	f_K	m_η	$m_{\eta'}$	θ_p
a	138*	494*	92*	114*	476	986	-14
b	138*	494*	92*	114*	487	958	-15
c	138*	494*	92*	114*	480	1020	-13
d	138*	494*	92*	114*	472	959	-15
e	138*	494*	92*	129	533	1097*	-1.2
f	138*	494*	92*	129	540	1097*	0.5

Table 3.
The characteristics of the light scalar nonet in MeV and the singlet-octet mixing angle θ_s in degrees.

	$m_{a_0} \sim a_0(980)$	$m_{K_0^*} \sim K_0^*(800)$	$m_\sigma \sim f_0(600)$	$m_{\sigma'} \sim f_0(980)$	θ_s
a	1040	1267	806	1438	24
b	981	1219	781	1427	24
c	1056	1280	805	1447	23.7
d	967	1208	762	1426	23.5
e	980*	1029	413	1123	19.5
f	980*	992	346	1073	18

Dimensionless parameter

$$\zeta = \frac{\kappa^2}{32G^3}\left(\frac{\lambda}{2\pi}\right)^4 . \tag{62}$$

(b): $\zeta^2/2 = 0.0096$, (d):$\zeta^2/2 = 0.021$, (f):$\zeta^2/2 = 0.0023$